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OBSERVATIONS ON IDENTITIES AND RELATIONS FOR INTERPOLATION FUNCTIONS AND SPECIAL NUMBERS

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ABSTRACT. The main motivation of this paper is to study and investigate a new family of combinatorial numbers with their generating functions. Firstly, we obtain some finite series representations including well-known numbers such as the Apostol-Bernoulli numbers, the Apostol-Euler numbers, a family of combinatorial numbers, the Daehee numbers, the Changhee numbers and the Stirling numbers of the second kind. Secondly, applying Mellin transform to these functions, we give interpolation functions for these numbers. We investigate some properties of these functions and other related complex valued functions. We observe that some special values of these functions give us the terms of some well-known infinite series. Thus, these functions unify the terms of some well-known identities and functions such as Hasse identity, the polylogarithm function, the digamma function, the Riemann zeta functions, the alternating Riemann zeta function, the Hurwitz zeta function, the alternating Hurwitz zeta function, the Hurwitz-Lerch zeta function and the other functions. Moreover, we give some remarks and observations about these functions related to some special numbers and polynomials such as the Stirling numbers of the second kind, the harmonic numbers, the array polynomials and also related to hypergeometric functions, the family of zeta functions. We also give not only Riemann integral representation, but also Cauchy integral representations for this new family of combinatorial numbers. Finally, in order to compute numerical values of these interpolation functions and other related complex valued functions, we present two algorithms. Furthermore, by using these algorithms, we provide some plots of these functions. Also, we investigate the effects of their parameters.

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1. INTRODUCTION

Interpolation functions for the special numbers and polynomials are very important in analytic number theory and mathematical pyhsics. The motivation of this paper is to derive interpolation functions for the combinatorial numbers $y_1(n,k;\lambda)$. We investigate some properties of these functions and other related complex valued functions. We also investigate relations

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between these functions and some well-known functions such as the polylogarithm function, the digamma function, the Riemann zeta functions, the alternating Riemann zeta function, the Hurwitz zeta function, the alternating Hurwitz zeta function, the Hurwitz-Lerch zeta function and the other functions. Moreover, we give some finite series representations including the Apostol-Bernoulli numbers, the Apostol-Euler numbers, the combinatorial numbers $y_1(n,k;\lambda)$, the Daehee numbers, the Changhee numbers and the Stirling numbers of the second kind.

In order to derive the results of this paper, we need the following notations, definitions, identities and formulas:

Let $\mathbb{N}=\{1,2,3,\ldots\}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Here, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ corresponds the set of integers, the set of real numbers and the set of complex numbers, respectively. Let tacitly assume that

$$0^n=\begin{cases}1, & n=0\\ 0, & n\in\mathbb{N}\end{cases}.$$

Simsek [33] defined the combinatorial numbers $y_1(n,k;\lambda)$ by means of the following generating function:

$$F_{y1}(t,k;\lambda)=\frac{1}{k!}\left(\lambda e^t+1\right)^k=\sum_{n=0}^{\infty}y_1(n,k;\lambda)\frac{t^n}{n!} \tag{1}$$

where $k\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$. From the above equation, one has the explicit formula for the combinatorial numbers $y_1(n,k;\lambda)$ as follows:

$$y_1(n,k;\lambda)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}j^n\lambda^j \tag{2}$$

where $n\in\mathbb{Z}^+$. Note that for $\lambda=1$, the numbers $y_1(n,k;\lambda)$ are reduced to the following combinatorial sum:

$$B(n,k)=k!y_1(n,k;1)=\sum_{j=0}^{k}\binom{k}{j}j^n \tag{3}$$

(*cf.* [30], [33], [31]).

As in [33], assuming that $\lambda\neq 0$, Table 1 includes some values of the numbers $y_1(n,k;\lambda)$ for $k=0,1,2,3,4$ and $n=0,1,2,3,4,5$:

$n\backslash k$	0	1	2	3	4
0	1	$\lambda+1$	$\frac{1}{2}\lambda^2+\lambda+\frac{1}{2}$	$\frac{1}{6}\lambda^3+\frac{1}{2}\lambda^2+\frac{1}{2}\lambda+\frac{1}{6}$	$\frac{1}{24}\lambda^4+\frac{1}{6}\lambda^3+\frac{1}{4}\lambda^2+\frac{1}{6}\lambda+\frac{1}{24}$
1	0	λ	$\lambda^2+\lambda$	$\frac{1}{2}\lambda^3+\lambda^2+\frac{1}{2}\lambda$	$\frac{1}{6}\lambda^4+\frac{1}{2}\lambda^3+\frac{1}{2}\lambda^2+\frac{1}{6}\lambda$
2	0	λ	$2\lambda^2+\lambda$	$\frac{3}{2}\lambda^3+2\lambda^2+\frac{1}{2}\lambda$	$\frac{2}{3}\lambda^4+\frac{3}{2}\lambda^3+\lambda^2+\frac{1}{6}\lambda$
3	0	λ	$4\lambda^2+\lambda$	$\frac{9}{2}\lambda^3+4\lambda^2+\frac{1}{2}\lambda$	$\frac{8}{3}\lambda^4+\frac{9}{2}\lambda^3+2\lambda^2+\frac{1}{6}\lambda$
4	0	λ	$8\lambda^2+\lambda$	$\frac{27}{2}\lambda^3+8\lambda^2+\frac{1}{2}\lambda$	$\frac{32}{3}\lambda^4+\frac{27}{2}\lambda^3+4\lambda^2+\frac{1}{6}\lambda$
5	0	λ	$16\lambda^2+\lambda$	$\frac{81}{2}\lambda^3+16\lambda^2+\frac{1}{2}\lambda$	$\frac{128}{3}\lambda^4+\frac{81}{2}\lambda^3+8\lambda^2+\frac{1}{6}\lambda$

Table 1. Some numerical values of the numbers $y_1(n,k;\lambda)$ (*cf.* [33]).

The Stirling numbers of the second kind $S_2(n,k)$ are defined by the following explicit formula:

$$S_2(n,v)=\frac{1}{v!}\sum_{j=0}^{v}\left(\begin{array}{c} v\\ j\end{array}\right)(-1)^j\,(v-j)^n \tag{4}$$

These numbers have the following relations: $S_2(0,0)=1$, $S_2(n,v)=0$ if $v>n$; $S_2(n,0)=0$ if $n>0$ (*cf.* [3], [6], [7], [9], [38]; and the references cited therein).

A relation between the Stirling numbers of the second kind and the combinatorial numbers $y_1(n,k;\lambda)$ given by (*cf.* [30]):

$$S_2(n,k)=(-1)^k\,y_1(n,k;-1)\,. \tag{5}$$

In [4, p.4, Eq-(7)], Boyadzhiev gave the following two identities related to the Stirling numbers of the second kind:

$$\sum_{k=0}^{n}\left(\begin{array}{c} n\\ k\end{array}\right)k^p x^k=\sum_{j=0}^{p}\left(\begin{array}{c} n\\ j\end{array}\right)S_2(p,j)\,j!x^j\,(1+x)^{n-j} \tag{6}$$

and

(7)

$$\sum_{k=0}^{n}\left(\begin{array}{c} n\\ k\end{array}\right)(-1)^{k-1}\,k^p x^k=\sum_{j=0}^{p}\left(\begin{array}{c} n\\ j\end{array}\right)S_2(p,j)\,j!\,(-1)^{j-1}\,x^j\,(1-x)^{n-j}\,.$$

It also should be note that

$$B(m,n)=\sum_{j=0}^{m}\left(\begin{array}{c} n\\ j\end{array}\right)j!2^{n-j}S_2(m,j) \tag{8}$$

(*cf.* [33]).

The λ-array polynomials $S_v^n(x;\lambda)$ are defined by Simsek [27] with the following generating function:

$$\frac{\left(\lambda e^t-1\right)^v}{v!}e^{xt}=\sum_{n=0}^{\infty}S_v^n(x;\lambda)\frac{t^n}{n!}$$

where $v\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$. For $\lambda=1$, these polynomials are reduced to the classical array polynomials $S_v^n(x)$:

$$S_v^n(x)=S_v^n(x;1)$$

which are given by the following combinatorial sum:

$$S_v^n(x)=\sum_{j=0}^{n}\left(\begin{array}{c} n\\ j\end{array}\right)x^{n-j}S_2(j,v) \tag{9}$$

(*cf.* [2], [5], [6], [26, Theorem 2], [27, Remark 3.3]).

A relation between the λ-array polynomials $S_v^n(x;\lambda)$ and the combinatorial numbers $y_1(n,k;\lambda)$ given by

$$S_k^n(0;\lambda)=(-1)^k\,y_1(n,k;-\lambda)\,.$$

The λ-Stirling numbers are defined by

$$S_2(n,v;\lambda)=\frac{1}{v!}\sum_{j=0}^{v}(-1)^{v-j}\left(\begin{array}{c} v\\ j\end{array}\right)\lambda^j j^n$$

(*cf.* [24], [27]). A relation between the λ-Stirling numbers and the combinatorial numbers $y_1(n,k;\lambda)$ given by

$$S_2(n,v;\lambda) = (-1)^v y_1(n,v;-\lambda). \tag{10}$$

The Bernstein basis functions $B_k^n(x)$ are defined as follows:

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k} \tag{11}$$

and integral formula for these polynomials are given as follows:

$$\int_0^1 B_k^n(x)\,dx = \frac{1}{n+1} \tag{12}$$

(*cf.* [23]).

The Apostol-Bernoulli numbers $\mathcal{B}_n(\lambda)$ are defined by

$$F_{\mathcal{B}}(t;\lambda) = \frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(\lambda)\frac{t^n}{n!}, \tag{13}$$

where $|t + \log\lambda| < 2\pi$ (*cf.* [38]).

The Apostol-Euler numbers $\mathcal{E}_n(\lambda)$ are defined by

$$F_{\mathcal{E}}(t;\lambda) = \frac{2}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(\lambda)\frac{t^n}{n!}, \tag{14}$$

where $|t + \log\lambda| < \pi$ (*cf.* [19], [38]; see also the references cited in each of these earlier works).

The Daehee numbers are defined by

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} \tag{15}$$

(*cf.* [15], [20], [22]; see also the references cited in each of these earlier works).

By using (15), one has the explicit formula for the Daehee numbers given by

$$D_n = (-1)^n \frac{n!}{n+1} \tag{16}$$

(*cf.* [25, p. 45], [20], [29]; see also the references cited in each of these earlier works).

The remainder of the present paper is summarized as follows:

In Section 1, we derive some finite series representations including the Apostol-Bernoulli numbers, the Apostol-Euler numbers, the combinatorial numbers $y_1(n,k;\lambda)$, the Daehee numbers, the Changhee numbers and the Stirling numbers of the second kind. We also give some remarks and corollaries.

In Section 2, we present an interpolation function of the combinatorial numbers $y_1(n,k;\lambda)$. Furthermore, we give some remarks and applications including complex valued functions associated with these interpolation functions and some special functions.

In Section 3, we present integral representations for the combinatorial numbers $y_1(n,k;\lambda)$ including not only Riemann integral, but also Cauchy integral representations.

In Section 4, we provide computation algorithms for the newly defined complex valued functions and their interpolation functions. We also provide some plots of these functions.

2. Finite series representations including the combinatorial numbers $y_1(n,k;\lambda)$, the Daehee numbers and the Changhee numbers

In this section, we give finite series representations including the Apostol-Bernoulli numbers, the Apostol-Euler numbers, the combinatorial numbers $y_1(n,k;\lambda)$, the Daehee numbers, the Changhee numbers and the Stirling numbers of the second kind. We also give some remarks and corollaries.

From (1) and (14), we have

$$\frac{2}{k}F_{y1}(t,k-1;\lambda)=F_{y1}(t,k;\lambda)\,F_{\mathcal{E}}(t;\lambda)$$

which is the special case when $v=1$ of a functional equation given by Yuluklu et al. [39, p. 4842].

It follows from the above functional equation that

$$\frac{2}{k}\sum_{n=0}^{\infty}y_1(n,k-1;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}y_1(n,k;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{E}_n(\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the special case when $v=1$ of Theorem 2.8 in the work of Yuluklu et al. [39, p. 4842] as follows:

$$y_1(n,k-1;\lambda)=\frac{k}{2}\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_{n-j}(\lambda)\,y_1(j,k;\lambda). \tag{17}$$

Substituting $\lambda=1$ into (17) and combining (3) and (8) with final equation yields the following corollary:

Corollary 2.1.

$$B(n,k-1)=\frac{1}{2}\sum_{j=0}^{n}\sum_{v=0}^{j}\binom{n}{j}\binom{k}{v}v!2^{k-v}\mathcal{E}_{n-j}(1)\,S_2(j,v). \tag{18}$$

From (13) and (14), we have

$$\sum_{n=0}^{\infty}\mathcal{E}_n(-\lambda)\frac{t^n}{n!}=-2\sum_{n=0}^{\infty}\frac{\mathcal{B}_{n+1}(\lambda)}{n+1}\frac{t^n}{n!}.$$

By using the above equation, one can easily get the following well-known relation between the Apostol-Bernoulli numbers and the Apostol-Euler numbers

$$\mathcal{E}_n(-\lambda)=-\frac{2}{n+1}\mathcal{B}_{n+1}(\lambda);\qquad(\lambda\neq1). \tag{19}$$

Substituting the above equation into (17) yields the following theorem:

Theorem 2.2.

$$y_1(n,k-1;\lambda) = -k\sum_{j=0}^{n}\binom{n}{j}\frac{\mathcal{B}_{n-j+1}(-\lambda)}{n-j+1}y_1(j,k;\lambda). \tag{20}$$

Substituting $\lambda = 1$ into (20) and combining (3) and (8) with final equation yields the following corollary:

Corollary 2.3.

$$B(n,k-1) = -\sum_{j=0}^{n}\sum_{v=0}^{j}\binom{n}{j}\binom{k}{v}\frac{v!2^{k-v}\mathcal{B}_{n-j+1}(-1)S_2(j,v)}{n-j+1}. \tag{21}$$

From the work of Simsek [34, Corollary 2], we know that

$$\mathcal{B}_m(\lambda) = \frac{m}{2}\sum_{v=0}^{m-1}\frac{(-1)^{v+1}}{\lambda^v}Ch_v(-\lambda,1)S_2(m-1,v) \tag{22}$$

where $Ch_v(\lambda,1)$ denotes the Changhee type numbers which, for $\lambda = 1$, yields the Changhee numbers with

$$Ch_v = Ch_v(1,1) = \frac{(-1)^v v!}{2^v},$$

(*cf.* [21]).

Substituting (22) into (20) yields the following theorem:

Theorem 2.4.

$$\begin{aligned} y_1(n,k-1;\lambda) &= -\frac{k}{2}\sum_{j=0}^{n}\binom{n}{j}y_1(j,k;\lambda)\sum_{v=0}^{n-j}\frac{(-1)^{v+1}}{\lambda^v} \\ &\quad \times Ch_v(\lambda,1)S_2(n-j,v). \end{aligned} \tag{23}$$

Substituting $\lambda = 1$ into (24) and using (3) yields the following corollary:

Corollary 2.5.

$$\begin{aligned} B(n,k-1) &= -\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}B(j,k)\sum_{v=0}^{n-j}\frac{(-1)^{v+1}}{\lambda^v}Ch_v \\ &\quad \times S_2(n-j,v). \end{aligned} \tag{24}$$

Combining (16) and (19), we have

$$\mathcal{E}_n(-\lambda) = \frac{2(-1)^{n+1}D_n}{n!}\mathcal{B}_{n+1}(\lambda); \qquad (\lambda \neq 1). \tag{25}$$

Combining the above relation with (17) yields the following theorem:

Theorem 2.6.

$$\begin{aligned} y_1(n,k-1;\lambda) &= k\sum_{j=0}^{n}(-1)^{n-j+1}\binom{n}{j}\frac{D_{n-j}}{(n-j)!} \\ &\quad \times \mathcal{B}_{n-j+1}(-\lambda)y_1(j,k;\lambda). \end{aligned} \tag{26}$$

Substituting $\lambda = -\lambda$ into (26) and using (10) yields the following corollary:

Corollary 2.7.

$$S_2(n,k-1;\lambda) = -k\sum_{j=0}^{n}(-1)^{n-j+1}\binom{n}{j}\frac{D_{n-j}}{(n-j)!} \times \mathcal{B}_{n-j+1}(\lambda)\,S_2(j,k;\lambda). \tag{27}$$

3. Interpolation Function of the combinatorial numbers $y_1(n,k;\lambda)$

In this section, we give an interpolation function of the numbers $y_1(n,k;\lambda)$. This function interpolates the numbers $y_1(n,k;\lambda)$ at negative integers. Also, we investigate some properties of complex valued functions associated with these interpolation functions. Moreover, we give remarks, observations and applications on these functions.

Let $s\in\mathbb{C}$. By applying the Mellin transformation to (1), we get

$$g(s,k;\lambda) = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}F_{y1}(-t,k;\lambda)\,dt, \tag{28}$$

where $\Gamma(s)$ denotes the Euler gamma function and $Re(s)>0$.

Theorem 3.1. *Let $n\in\mathbb{Z}^+$. Then we have*

$$g(-n,k;\lambda) = y_1(n,k;\lambda).$$

Proof. The proof of this theorem is similar to that of Theorem 8 in [37, p. 254]. We give sketch of proof as follows: Substituting $s=-n$, $n\in\mathbb{Z}^+$ into (28) and by using Cauchy Residue Theorem in (28), we arrive at desired result. □

We modify integral equation in (28) as follows:
Let $\min\{Re(s), Re(x)\}>0$. Then,

$$\frac{1}{\Gamma(s)}\int_0^{\infty} F_{y1}(-t,k;\lambda)\,e^{-xt}t^{s-1}dt = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^j}{(j+x)^s} \tag{29}$$

where the additional constraint $Re(x)>0$ is required for the convergence of the infinite integral occurring on the left-hand side at its upper terminal.

By using the integral transform in (29), we define the following complex valued functions:

Definition 3.2. *Let $k\in\mathbb{N}$ and $s\in\mathbb{C}$. We define*

$$h(s,x;k;\lambda) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^j}{(j+x)^s}. \tag{30}$$

Note that there is one complex valued function for each value of k.
Setting $s=-n$ $(n\in\mathbb{Z}^+)$ into the above equation, we have

$$h(-n,x;k;\lambda) = \frac{1}{k!}\sum_{l=0}^{n}\binom{n}{l}x^{n-l}\sum_{j=0}^{k}\binom{k}{j}\lambda^j j^l.$$

By using (2) in the above equation, we arrive at the following theorem:

Theorem 3.3. *Let $n, k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. Then we have*

$$h(-n, x; k; \lambda) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} y_1(l, k; \lambda). \tag{31}$$

The right-hand side of the equation (31) is related to very famous combinatorial sums and special numbers. Therefore, we give some important applications for the formula in (31) as follows:

Substituting $\lambda = -1$ into (31) and using (5) and (9), we get a relation between the function $h(-n, x; k; \lambda)$ and the classical array polynomials by the following corollary:

Corollary 3.4. *Let $n, k \in \mathbb{N}$. Then we have*

$$h(-n, x; k; -1) = (-1)^k S_k^n(x). \tag{32}$$

Remark 3.5. *In the special case when $x = 1$, equation (32) is reduced to*

$$h(-n, 1; k; -1) = (-1)^k \sum_{l=0}^{n} \binom{n}{l} S_2(l, k).$$

Moreover, substituting $\lambda = 1$ into (31) and using (3), we also get the following corollary:

Corollary 3.6. *Let $n, k \in \mathbb{N}$. Then we have*

$$h(-n, x; k; 1) = \frac{1}{k!} \sum_{l=0}^{n} \binom{n}{l} x^{n-l} B(l, k). \tag{33}$$

Note that the right-hand side of the equation (30) is related to very famous special functions, combinatorial sums and special numbers. For this reason, in addition to the above applications, we also give some important applications for the formula in (30) as follows:

Remark 3.7. *Substituting $\lambda = -1$ and $s = 1$ into the equation (30), we get*

$$h(1, x; k; -1) = \frac{1}{x(x+1)\dots(x+k)} \tag{34}$$

which is deduced from the partial fraction decomposition given in [17, p.188] *for $k \in \mathbb{N}_0$ as follows:*

$$\frac{k!}{x(x+1)\dots(x+k)} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{j+x}; \qquad (x \notin \{0, -1, \dots, -n\}).$$

By letting $x \to -1$ in the above equation, we have

$$\sum_{j=2}^{k} (-1)^j \binom{k}{j} \frac{1}{j-1} = 1 - k + kH_{k-1} \ (k \in \mathbb{N})$$

where

$$H_k = \sum_{j=1}^{k} \frac{1}{j}$$

(cf. [8, Eq. (2.1)], [17, p.188] , [28] *and* [36]*).*

Remark 3.8. *In* [28], *Simsek proved the following combinatorial sum:*

$$\sum_{j=k}^{n}(-1)^{n-j}\left(\begin{array}{c}n\\j\end{array}\right)\frac{1}{j+1}=(-1)^{n-k}\left(\begin{array}{c}n\\k\end{array}\right)\frac{1}{n+1}.$$

Taking into account the above formula for $k=0$ and substituting $s=x=1$ and $\lambda=-1$ into (30), *then we also have*

$$k!h\left(1,1;k;-1\right)=\frac{1}{n+1}.$$

Remark 3.9. *If we substitute $s=x=\lambda=1$ into the equation* (30), *we have*

$$k!h\left(1,1;k;1\right)=\sum_{j=0}^{k}\left(\begin{array}{c}k\\j\end{array}\right)\frac{1}{j+1}=\frac{2^{k+1}-1}{k+1}$$

which was given in [36, Identity 13].

Remark 3.10. *From the work of Connon* [13, Eq-(4.4.99d)], *we have*

$$h\left(s,x;k;-1\right)=\frac{(-1)^{1-s}}{k!\left(s-1\right)!}\int_{0}^{1}t^{x-1}\left(1-t\right)^{k}\log^{s-1}tdt.$$

Remark 3.11. *Also, from the work of Connon* [11, p.62, Eq-(4.3.70a)], *we have*

$$\sum_{k=0}^{\infty}\frac{k!h\left(s,x;k;-1\right)}{k+1}=\frac{(-1)^{1-s}}{(s-1)!}\psi^{(s)}\left(x\right)$$

where $\psi^{(s)}\left(x\right)$ denotes s-th derivative of the digamma function, also called polygamma function defined by

$$\psi\left(x\right)=\frac{d}{dx}\left\{\log\Gamma\left(x\right)\right\}$$

(cf. [38, p. 24]*).*

Remark 3.12. *Observe that the function $h\left(s,x;k;\lambda\right)$ is related to the sum $S_k(\lambda,x)$ which was defined by Connon in* [12, p.132, Eq-(4.4.43b)] *and* [14, p.12, Eq-(4)]. *That is,*

$$S_k(\lambda,x)=k!h\left(s,x;k;\lambda\right).$$

We set

$$H\left(s,x;\lambda\right)=\sum_{k=0}^{\infty}a_k h\left(s,x;k;\lambda\right). \tag{35}$$

Substituting $a_k=(-1)^k D_k=\frac{k!}{k+1}$ and $\lambda=-\alpha$ into (35) reduces to the following identity, which was given by Connon [14, p. 15, Eq. (11)], we have

$$\sum_{k=0}^{\infty}(-1)^k h\left(s,x;k;-\alpha\right)D_k=s\Phi\left(\alpha,s+1,x\right)-\Phi\left(\alpha,s,x\right)\log\left(\alpha\right)$$

where $\Phi\left(\alpha,s,x\right)$ denotes the Hurwitz-Lerch zeta function which is defined by (*cf.* [38, p. 194 et seq.]):

$$\Phi\left(\alpha,s,x\right)=\sum_{n=0}^{\infty}\frac{\alpha^n}{\left(n+x\right)^s}$$

where $x \in \mathbb{C} \backslash \mathbb{Z}_0^-$; $s \in \mathbb{C}$ when $|\alpha| < 1$; $\mathrm{Re}(s) > 1$ when $|\alpha| = 1$. Some special cases of parameter α give us the following well-known interpolation functions for the Bernoulli numbers and polynomials such as the Riemann zeta function and the Hurwitz zeta function respectively as follows (see, for details, [38, Chapter 2]):

$$\Phi(1, s, 1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, (\mathrm{Re}(s) > 1)$$

and

$$\Phi(1, s, a) = \zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, (\mathrm{Re}(s) > 1).$$

Substituting $a_k = \frac{k!}{2^{k+1}} = \frac{(-1)^k Ch_k}{2}$, $\lambda = -1$ and $x = 1$ into (35) reduces to the following identity, which was given by Connon [10, p. 28, Eq. (3.11)], we have

$$\sum_{k=0}^{\infty} (-1)^k h(s, 1; k; -1) Ch_k = \left(1 - 2^{1-s}\right) \zeta(s).$$

The above equation is also known as Hasse's formula(may be also called Hasse-Sondow identity) proved and investigated before by Hasse and Sondow in [18] and [35].

Remark 3.13. *The sum in* (35) *is related to many known functions and also other unknown functions for the special values of the sequences* (a_k); $(k \in \mathbb{N})$. *That is, the special cases of the sequences* (a_k) *with different values of the parameters* λ *and* x *in* (35), *one has more identities for some infinite series related to the Hasse identity, the polylogarithm function, the digamma function, the Riemann zeta functions, the alternating Riemann zeta function, the Hurwitz zeta function, the alternating Hurwitz zeta function, the Hurwitz-Lerch zeta function and the other functions. For examples, the reader can consult the following references:* [1], [10], [11], [12], [13], [14, p. 12, Eq. (4)] *and* [38]; *(see also the related references cited therein).*

Now, we give a hypergeometric function representation of the function $h(s, x; k; \lambda)$ by the following theorem:

Theorem 3.14.

$$h(s, x; k; \lambda) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda^j x^{-s} {}_1F_0\left(-s; -; \frac{j}{x}\right)$$

where ${}_1F_0$ *denotes hypergeometric function defined by*

$${}_1F_0(a; -; x) = \sum_{n=0}^{\infty} (a)_n \frac{x^n}{n!}$$

where $(a)_n$ *denotes the falling factorial polynomials, which defined by*

$$(a)_n = a(a-1)(a-2)\ldots(a-n+1) \qquad (a \in \mathbb{C}; n \in \mathbb{N})$$

and

$$(a)_0 = 1 \qquad (a \in \mathbb{C}).$$

Proof. By using negative binomial series expansion in (30), we get

$$h(s,x;k;\lambda)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^{j}x^{-s}\sum_{n=0}^{\infty}\binom{-s}{n}\left(\frac{j}{x}\right)^{n},$$

Thus, we get

$$h(s,x;k;\lambda)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^{j}x^{-s}\sum_{n=0}^{\infty}(-s)_{n}\frac{\left(\frac{j}{x}\right)^{n}}{n!}.$$

By using definition of ${}_1F_0$ in the above equation, we arrive at the desired result. □

4. Integral representation for the numbers $y_1(n,k;\lambda)$

In this section, we give two type integral representations for the numbers $y_1(n,k;\lambda)$ including Riemann integral and Cauchy integral representations.

Firstly, we give the Riemann integral representation for the numbers $y_1(n,k;\lambda)$ with respect to λ. Integrating both sides of the equation (2), we get

$$\int_0^x y_1(n,k;\lambda)\,d\lambda=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}j^{n}\int_0^x\lambda^{j}d\lambda.$$

From the above equation, we arrive at the following lemma:

Lemma 4.1.

$$\int_0^x y_1(n,k;\lambda)\,d\lambda=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}j^{n}\frac{x^{j+1}}{j+1}. \tag{36}$$

Substituting $x=-1$ into (36), we arrive at the following corollary:

Corollary 4.2.

$$\int_{-1}^0 y_1(n,k;\lambda)\,d\lambda=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\frac{j^{n}}{j+1}. \tag{37}$$

Theorem 4.3.

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\frac{j^{n}}{j+1}=\frac{1}{k+1}\sum_{j=0}^{n}S_2(n,j)\,j!\,(-1)^{j}. \tag{38}$$

Proof. By combining (7) and (11), we have

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}j^{n}x^{j}=\sum_{j=0}^{n}S_2(n,j)\,j!\,(-1)^{j}B_j^{k}(x) \tag{39}$$

where $B_j^k(x)$ denotes the Bernstein basis functions. Integrating both sides of equation (39) from 0 to 1 and using equation (12), we get the desired result. □

Combining (37) with (38), we get the following corollary:

Corollary 4.4.

$$\int_{-1}^{0} y_1(n,k;\lambda)\,d\lambda = \frac{1}{(k+1)!}\sum_{j=0}^{n} S_2(n,j)\,j!\,(-1)^j . \tag{40}$$

By using same method that of [32], secondly we give a Cauchy integral representation for the numbers $y_1(n,k;\lambda)$. By applying the Cauchy Residue Theorem to the generating function for the numbers $y_1(n,k;\lambda)$, we derive a Cauchy integral representation for these numbers by the following theorem:

Theorem 4.5.

$$y_1(n,k;\lambda) = \frac{n!}{2\pi i}\int_{\mathcal{C}} F_{y1}(z,k;\lambda)\,\frac{dz}{z^{n+1}} \tag{41}$$

where $\mathcal{C}$ is a circle around the origin and the integration is in positive direction, $\lambda, z \in \mathbb{C}$ and $n \in \mathbb{Z}^+$.

Proof. By using Cauchy Residue Theorem with generating functions for the numbers $y_1(n,k;\lambda)$, we have

$$\frac{n!}{2\pi i}\int_{\mathcal{C}} F_{y1}(z,k;\lambda)\,\frac{dz}{z^{n+1}} = \frac{n!}{2\pi i}\left(2\pi i Res\left(\frac{F_{y1}(z,k;\lambda)}{z^{n+1}},0\right)\right), \tag{42}$$

where $Res(f(z),a)$ denotes the residue of $f(z)$ function at $z=a$. By using the following Laurent series expansion,

$$\begin{aligned}\frac{F_{y1}(z,k;\lambda)}{z^{n+1}} &= \sum_{m=0}^{\infty} y_1(m,k;\lambda)\,\frac{z^{m-n-1}}{m!}\\ &= y_1(0,k;\lambda)\,\frac{1}{z^{n+1}} + y_1(1,k;\lambda)\,\frac{1}{z^n} + \cdots + \frac{y_1(n-1,k;\lambda)}{(n-)\,1!}\frac{1}{z^2}\\ &\quad + \frac{y_1(n,k;\lambda)}{n!}\frac{1}{z} + \frac{y_1(n+1,k;\lambda)}{(n+1)!} + \cdots .\end{aligned}$$

we compute residue of the function $\frac{1}{z^{n+1}}F_{y1}(z,k;\lambda)$ at $z=0$ as follows:

$$Res\left(\frac{F_{y1}(z,k;\lambda)}{z^{n+1}},0\right) = \frac{y_1(n,k;\lambda)}{n!}.$$

Thus, by combining the above equation with (42), we arrive at the desired result. □

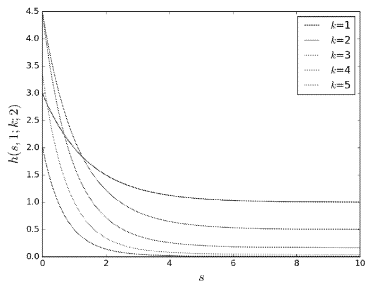
5. Computations and Numerical Evaluations

In this section, in order to compute numerical values of the functions $h(s,x;k;\lambda)$ given in (30) and interpolation functions $h(-n,x;k;\lambda)$ given in (31) related to the combinatorial numbers $y_1(n,k;\lambda)$, we present two algorithms. Furthermore, by using these algorithms, we provide some plots of the function $h(s,x;k;\lambda)$ for some special values of its parameters. By means of the drawn plots, the effects of s, x, k and λ are demonstrated. Also, we present some observations and evaluations related to effects of the parameters.

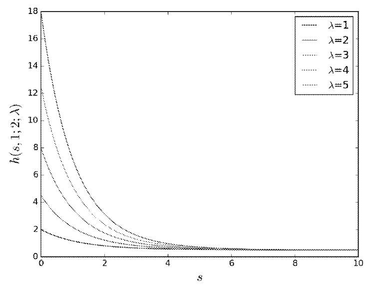
Algorithm 1 Let $k \in \mathbb{N}$ and $s \in \mathbb{C}$. This algorithm will return numerical values of the function $h(s, x; k; \lambda)$ given in (30).

procedure H_FUNC(s, x, k, λ)
    Begin
    Lobal variables:
    $j \leftarrow 0, S \leftarrow 0$
    for all j in $\{0, 1, 2, \ldots, k\}$ **do**
        $S \leftarrow S + \text{Binomial\_Coef}(k, j) * (\text{Power}(\lambda, j) / \text{Power}(j + x, s))$
    end for
    return $(1/\text{Factorial}(k)) * S$
end procedure

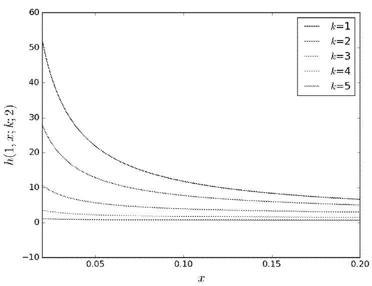
By using Algorithm 1, we draw a few graphs of the function $h(s, x; k; \lambda)$ for some special cases of its parameters and Figure 1 shows these graphs.



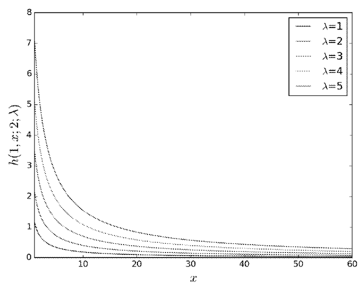
(a) The function $h(s, x; k; \lambda)$ for $x = 1$, $\lambda = 2$, $k \in \{1, 2, 3, 4, 5\}$ and $s \in [0, 10]$.



(b) The function $h(s, x; k; \lambda)$ for $x = 1$, $k = 2$, $\lambda \in \{1, 2, 3, 4, 5\}$ and $s \in [0, 10]$.



(c) The function $h(s, x; k; \lambda)$ for $s = 1$, $\lambda = 2$, $k \in \{1, 2, 3, 4, 5\}$ and $x \in \left[0, \frac{1}{5}\right]$.



(d) The function $h(s, x; k; \lambda)$ for $s = 1$, $k = 2$, $\lambda \in \{1, 2, 3, 4, 5\}$ and $x \in [0, 60]$.

Figure 1. Some plots of the function $h(s, x; k; \lambda)$ for some special values of its parameters.

Algorithm 2 Let $n, k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. This algorithm will return numerical values of the interpolation functions $h(-n, x; k; \lambda)$ given in (31) related to the combinatorial numbers $y_1(n, k; \lambda)$ whose computation algorithm was given in [33, Algorithm 1] by Simsek.

procedure H_INT_FUNC($-n$, x, k, λ)
  Begin
  Lobal variables:
  $l \leftarrow 0, S \leftarrow 0$
  for all l in $\{0, 1, 2, \ldots, n\}$ **do**
    $S \leftarrow S + \text{Binomial\_Coef}(n, l) * \text{Power}(x, n-l) * y_1(l, k; \lambda)$
  end for
  return S
end procedure

By the help of Algorithm 1 and Algorithm 2, we draw the functions $h(s, x; k; \lambda)$ and the interpolation functions $h(-n, x; k; \lambda)$ for $s = -n$; $n \in \mathbb{Z}^+$ in the case of $x = 1$, $k = 1, 2, 3, 4, 5$ and $\lambda = 2$, and these graphs are given by Figure 2.

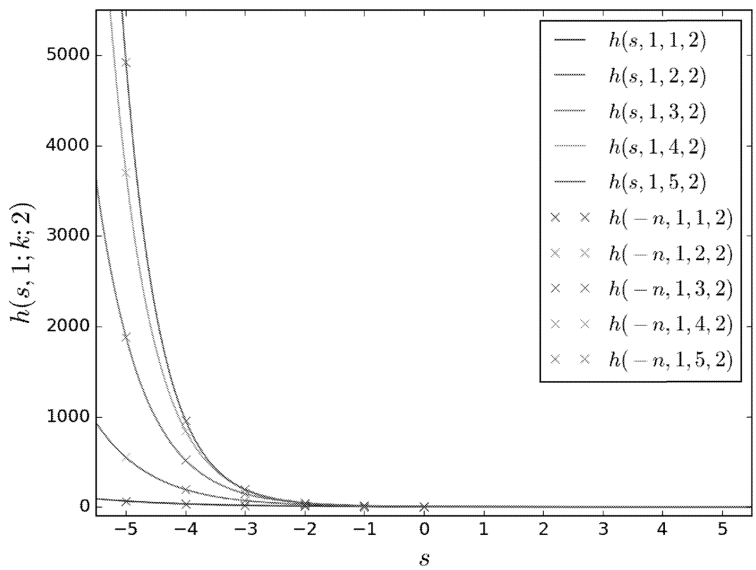


Figure 2. Combined graphs of the functions $h(s, x; k; \lambda)$ and the interpolation functions $h(-n, x; k; \lambda)$ for $s = -n$; $n \in \mathbb{Z}^+$, in the case of $x = 1$, $k = 1, 2, 3, 4, 5$ and $\lambda = 2$.

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